A REMARK ON EIGENVALUES OF CERTAIN POSITIVE OPERATORS

BY

I. GLICKSBERG*

ABSTRACT

For a transition operator on $C(X)$ with an invariant measure having global support, the point spectrum of unit modulus and the corresponding eigenfunctions are related to the action of a compact group on a quotient space of X.

The purpose of this paper is to observe that the results of an earlier note [1] apply to the situation considered by Rota [2] or its more general setting due to Schaefer [3]. Rota showed that if T is a non-negative operator of norm $\leqq 1$ on both L_1 and L_∞ of a finite measure space, and α is an eigenvalue of unit modulus, $Tf = \alpha f$, $0 \neq f \in L_1$, then α^2 is an eigenvalue (with $T|f|\,g^2 = \alpha^2 |f|\,g^2$ where $f = |f| \cdot g$). Using relevant portions of Rota's argument, Schaefer [3] showed that if T is a non-negative operator of spectral radius $\leqq 1$ on a space E which is either an L_p space of any measure space, $1 \leqq p < \infty$, or a $C(X)$, while $T^*\psi \leqq \psi$ for some continuous linear functional on E which is strictly positive on the non-negative cone (less 0), then $Tf = \alpha f$, $|\alpha| = 1$, $0 \neq f \in E$ implies $T\,|f|\,g^n = \alpha^n |f|\,g^n$ for all n.

Our observation borrows a portion of Rota's proof not used by Schaefer as well as part of Schaefer's, and applies in his setting (with our T his operator U [3, p. 58]).

Suppose Y is compact Hausdorff and T is a non-negative operator on $C(Y)$ with $T1 = 1$ while(1) *$T^*\mu = \mu$ for some probability measure with (minimal) support all of Y. Then the closed span A of the eigenfunctions of T with eigenvalues of unit modulus is a self adjoint subalgebra of $C(Y)$, and thus isomorphic*

Received December 26, 1968.

* Work supported in part by the National Science Foundation.

(1) This is no less general than $T^*\mu \leq \mu$ since when $T1 = 1$, $\|T^*\mu\| = T^*\mu(1) = \mu(1) = \|\mu\|$ and this implies $T^*\mu = \mu$. That T^* has a fixed point in the probability measures is of course no restriction; and if $T^*\mu = \mu$ and F is the support of μ we can always restrict our attention to F: indeed if $h \in C(Y)$, $h = 0$ on F implies $\mu\,(T(|h|)) = 0$ so $|Th| \leqq T\,|h| = 0$, and thus T yields a well defined map T_1 of $C(F)$ into itself. But of course our conclusions are then only about T_1 and its eigenfunctions. (More generally, F could be the closure of the union of the supports of all fixed μ).

to $C(X)$ for some quotient space X of Y; moreover there is a compact monothetic group K with generator $k_0 \in K$ which acts as a transformation group on X, for which

$$Th(x) = h(k_0 x), \qquad h \in C(X), \qquad x \in X, \tag{1}$$

when $T \mid A$ is viewed as an operator on $C(X)$. Finally identifying the character group $K^{\wedge}$ of K with a subgroup of the circle group, the eigenvalues of unit modulus of T are just the elements of $\bigcup_{x \in X} H_x^{\perp}$, where H_x is the isotropy subgroup of x, $H_x^{\perp}$ the orthogonal subgroup.

For Schaefer's operator U, as a functional μ is just a multiple of $h \to \psi(|f|h)$ The fact that A is a subalgebra is simple: we have $Th(y) = \int h dm_y$, $h \in C(Y)$, for some probabability measure m_y, and (Rota) if $Th = \alpha h$, $|\alpha| = 1$, then

$$|\int h dm_y|^2 = | Th(y)|^2 = |h(y)|^2 \leqq \int |h|^2 dm_y = T|h|^2(y) \tag{2}$$

by Schwarz, so $T|h|^2 - |h|^2 \geqq 0$. But (Schaefer) $\mu(T|h|^2 - |h|^2) = (T^*\mu - \mu|$ $(|h|^2) = 0$, and since $T|h|^2 - |h|^2 \geqq 0$ while $\mu \geqq 0$ has global support, $T|h|^2 = |h|^2$. Thus from equality in the Schwarz inequality (2) we have h constant a.e. m_y, with $\alpha h(y)$ the constant since $\int hdm_y = \alpha h(y)$. So if $Tf = \beta f$, $\beta| = 1$, then $Tfh(y) = \int fh\,dm_y = \alpha h(y) \int f dm_y = \alpha\beta fh(y)$, whence fh is an eigenfunction with eigenvalue of unit modulus, or zero, so A is an algebra. Trivially A is self-adjoint since $T\bar{h} = \overline{Th}$.

Now A appears as $C(X)$ as asserted, and since A is invariant under T we can view $T_0 = T|A$ as an operator on $C(X)$. A is spanned by eigenfunctions with eigenvalues of modulus 1, so the action of $\{T_0^n : n \geqq 1\}$ on A, or on $C(X)$, is almost periodic. Thus by (a simplification of) the result of [1] (noting that there $C_p = C(X)$, so E is the identity operator) we obtain our transformation group on X with generator k_0 satisfying (1). In fact [1], K is isomorphic to the kernel $\mathscr{K}$ of the semigroup Σ formed by the strong operator closure of $\{T_0^n : n \geqq 1\}$, and thus to Σ itself since $\mathscr{K} = E\Sigma = \Sigma$, E being the identity. Evidently each eigenfunction h of T_0 is a common eigenfunction of the elements of Σ (and conversely), and so of the action of the elements of K:

$$h(kx) = \chi(k)\, h(x),\ \chi(k) \in \mathbb{C}. \text{ (}^2\text{)} \tag{3}$$

χ is easily seen to be a character of K. Thus each common eigenfunction h of the action of the elements of K lies in $\bigcup_{\chi \in K^{\wedge}} P_\chi C(X)$, where P_χ is the projection on $C(X)$ defined by

$$P_\chi h(x) = \int_K \overline{\chi(k)}\, h(kx) dk \tag{4}$$

since (3) evidently implies $P_\chi h = h$. Conversely (4) implies (3), so our set of common eigenfunctions are the non-zero elements of $\bigcup_{\chi \in K^\wedge} P_\chi C(X)$.

To obtain the eigenvalues of unit modulus of T (= those of T_0) we must thus just identify $\{\chi(k_0) : \chi \in K^\wedge,\ P_\chi \neq 0\}$. Now each orbit Kx is homeomorphic to K/H_x so each $\chi \in H_x^\perp$ defines a continuous unimodular function $kx \to \chi(k)$ on Kx; let h be any continuous extension to all of X. Then $P_\chi\ h(x) = 1$, so $P_\chi \neq 0$· Conversely if $P_\chi\ h = h_1 \neq 0$ for some $h \in C(X)$ then if $h_1(x) \neq 0$, $h_1\ (kx) = \chi(k) \cdot h_1(x)$ whence $\chi(k) \cdot h_1(x) = h_1(x)$ and $\chi(k) = 1$ if $k \in H_x^\perp$ so $\chi \in H_x^2$. Thus our eigenvalues are precisely

$$\{\chi(K_0) : \chi \in \bigcup_{x \in X} H_x^\perp\}. \tag{5}$$

Modulo the injection of $K^\wedge$ into the circle group dual to the map $n \to n k_0$ of Z densely into K we can identify (5) with the subset $\bigcup H_x^\perp$ of $K^\wedge$ as asserted, completing our proof.

Note that if X contains two distinct orbits we can produce non-constant fixed points of T: with $h \equiv 1$ on one orbit and 0 on the other and continuous, $0 \neq P_1 h = T P_1 h$. Thus we have precisely one orbit when only the constants are left fixed by T, and then the unimodular eigenvalues form a group (since all H_x's coincide, K being abelian).

In the preceding no use was made of the invariant measure μ beyond showing A to be an algebra. But in the context of Schaefer's second result [3, Th.2] it shows that result as stated holds vacuously if $p < \infty$; more precisely the result should assert that *if T is a non-negative operator of spectral radius $\leqq 1$ on l_p, $1 \leqq p < \infty$, while $T^*\psi \leqq \psi$ for some strictly positive continuous linear functional ψ on l_p then all eigenvalues of T of unit modulus are roots of unity.*

Indeed such an eigenvalue of T is an eigenvalue of a corresponding operator U as in [3, p. 58]: if $Tf = \alpha f$, $|\alpha| = 1$, $f \neq 0$, then $T|f| = |f|$, and restricting our attention to the support of f, which we may as well take as the entire set of non-negative integers Z_+

$$Uh = |f|^{-1}\, T|f| h, \quad h \in l_\infty,$$

(2) With $S \leftrightarrow k$ for $S \in \mathscr{K}$, $k \in K$ iff $Sh(x) = h(kx)$, $x \in X$, $h \in C(X)$.

defines a non-negative operator with $U1 = 1$; and then $\psi(|f|\ Uh) - \psi(|f|\,h) = \psi(T(|f|\,h)) - \psi(|f|\,h) \leqq 0$ for $0 \leqq h \in l_\infty$. Since $\psi \in l_p^* = l_q$ is strictly positive $h \to \mu(h) = \psi(|f|\,h)$, $h \in l_\infty$, is given by a non-negative measure with support all of Z_+. Thus U provides an operator on $C(\beta(Z_+))$ of the type required in our result, but with a purely atomic invariant [3] measure μ. Let ρ map $Y = \beta(Z_+)$ onto its quotient X. Then ρ also carries μ into an invariant measure ν which is of course still purely atomic: $\nu(x\} = \Sigma_{y \in \rho^{-1}(x)}\mu\{y\}$, and is carried by the countable image of Z_+. Each such image point x has a finite orbit Kx since $\nu\{k \cdot x\} = \nu\{x\}$. Thus $\{U^n\chi_{\rho^{-1}(x)}: n \geqq 1\} = \{\chi_{\rho^{-1}(k_0^n x)} : n \geqq 1\}$ is a finite collection; on the other hand Schaefer shows that if α is not a root of unity, for every integer n there is a non-void $F_n \subset Z_+$ with $U\chi_{F_{n+1}} = \chi_{F_n}$, $F_n \cap F_m = \emptyset$, $n \neq m$. Since F_n is defined (in Z_+) by an eigenfunction in A, F_n is a union of sets $\rho^{-1}(x) \cap Z_+$, whence $F_n \cap F_{n+m} \neq \emptyset$ for some $m > 0$, a contradiction showing α must be a root of unity.

References

1. I. Glicksberg, *A remark on almost periodic transition operators*, Israel J. Math. **3** (1965), 71–74.

2. G.-C. Rota, *On the eigenvalues of positive operators*, Bull. Amer. Math. Soc. **67** (1961), 556–558.

3. H. H. Schaefer, *On the point spectrum of positive operators*, Proc. Amer. Math. Soc. **15** (1964), 56–60.

University of Washington

(3) $U^* \mu \leqq \mu$ and thus $U^* \mu = \mu$; cf. footnote (1).